

A MARCINKIEWICZ MAXIMAL-MULTIPLIER THEOREM

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ABSTRACT. For $r < 2$, we prove the boundedness of a maximal operator formed by applying all multipliers m with $\|m\|_{V^r} \leq 1$ to a given function.

1. INTRODUCTION

Given an exponent r and a function f defined on \mathbb{R} , consider the r -variation norm

$$\|f\|_{V^r} = \|f\|_{L^\infty} + \sup_{N, \xi_0 < \dots < \xi_N} \left(\sum_{i=1}^N |f(\xi_i) - f(\xi_{i-1})|^r \right)^{1/r}$$

where the supremum is over all strictly increasing finite length sequences of real numbers.

The classical Marcinkiewicz multiplier theorem states that if $r = 1$ and a function m is of bounded r -variation uniformly on dyadic shells, then m is an L^p multiplier for $1 < p < \infty$ and

$$(1) \quad \|(m\hat{f})^\sim\|_{L^p} \leq C_{p,r} \sup_{k \in \mathbb{Z}} \|1_{D_k} m\|_{V^r} \|f\|_{L^p}$$

where $D_k = [-2^{k+1}, -2^k) \cup (2^k, 2^{k+1}]$ and $\hat{\cdot}, \sim$ denote the Fourier-transform and its inverse. Later, Coifman, Rubio de Francia, and Semmes [2], see also [8], showed that the requirement of bounded 1-variation can be relaxed to allow for functions of bounded 2-variation, and in fact (1) holds whenever $r \geq 2$ and $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{r}$.

The estimate [2] does not discriminate between multipliers of bounded 2-variation and those of bounded r -variation where $r < 2$, and so one might ask whether there is anything to be gained by controlling the variation norm of multipliers in the latter range of exponents.

Defining the maximal-multiplier operator

$$(2) \quad \mathcal{M}_r[f](x) = \sup_{m: \|m\|_{V^r} \leq 1} |(m\hat{f})^\sim(x)|$$

where the supremum is over all functions in the V^r unit ball, we have

Theorem 1.1. *Suppose $1 \leq r < 2$ and $r < p < \infty$. Then*

$$(3) \quad \|\mathcal{M}_r[f]\|_{L^p} \leq C_{p,r} \|f\|_{L^p}.$$

The case $r = 1$ was observed independently by Lacey [4].

Note that, in the definition of \mathcal{M}_r , each m is required to have finite r -variation on all of \mathbb{R} rather than simply on each dyadic shell as in (1). This is necessary for boundedness, as can be seen from the counterexamples of Christ, Grafakos, Honzík and Seeger [1].

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Although the maximal operator (2) would seem to be fairly strong, we do not yet know of an application for the bound above. We will, however, quickly illustrate a strategy for its use that falls an (important) ϵ short of success. Let Ψ be (say) a Schwartz function, and for each $\xi, x \in \mathbb{R}$ and $k \in \mathbb{Z}$ consider the 2^k -truncated partial Fourier integral

$$\mathcal{S}_k[f](\xi, x) = p.v. \int f(x-t) e^{2\pi i \xi t} \Psi(2^{-k}t) \frac{1}{t} dt.$$

It was proven by Demeter, Lacey, Tao, and Thiele [3] that for $q = 2$ and $1 < p < \infty$

$$(4) \quad \left\| \sup_{\|g\|_{L^q}=1} \left\| \sup_k |(\mathcal{S}_k[f](\cdot, x) \hat{g})^\vee| \right\|_{L^q} \right\|_{L_x^p} \leq C_{p,q} \|f\|_{L^p}.$$

If we had the bound

$$(5) \quad \|\mathcal{S}_k[f](\xi, x)\|_{L_x^p(\ell_k^\infty(V_\xi^r))} \leq C_{p,r} \|f\|_{L^p}$$

for some $r < 2$, then an application of Theorem 1.1 would give (4) for $q > r$ by rather different means than [3]. In fact, one can see by applying the method in Appendix D of [6] that (5) holds for $r > 2$ and $p > r'$. Unfortunately, it does fail for $r \leq 2$.

2. PROOF OF THEOREM 1.1

The following lemma was proven in [2], see also [5].

Lemma 2.1. *Let m be a compactly supported function on \mathbb{R} of bounded r -variation for some $1 \leq r < \infty$. Then for each integer $j \geq 0$, one can find a collection Υ_j of pairwise disjoint subintervals of \mathbb{R} and coefficients $\{b_v\}_{v \in \Upsilon_j} \subset \mathbb{R}$ so that $|\Upsilon_j| \leq 2^j$, $|b_v| \leq 2^{-j/r} \|m\|_{V_r}$, and*

$$(6) \quad m = \sum_{j \geq 0} \sum_{v \in \Upsilon_j} b_v 1_v$$

where the sum in j converges uniformly.

The lemma above was applied in concert with Rubio de Francia's square function estimate [7] to obtain (1). Here, we will argue similarly, exploiting the analogy between the Rubio de Francia square function estimate and the variation-norm Carleson theorem.

It was proven in [7] that for $p \geq 2$

$$\sup_{\mathcal{I}} \left\| \left(\sum_{I \in \mathcal{I}} |(1_I \hat{f})^\vee|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \|f\|_{L^p}$$

where the supremum above is over all collections of pairwise disjoint subintervals of \mathbb{R} . Consider the partial Fourier integral

$$\mathcal{S}[f](\xi, x) = (1_{(\infty, \xi]} \hat{f})^\vee(x).$$

It was proven in [6] that for $s > 2$ and $p > s'$

$$\|\mathcal{S}[f](\xi, x)\|_{L_x^p(V_\xi^s)} \leq C_{p,s} \|f\|_{L^p}$$

or, equivalently,

$$(7) \quad \left\| \sup_{\mathcal{I}} \left(\sum_{I \in \mathcal{I}} |(1_I \hat{f})^\vee|^s \right)^{1/s} \right\|_{L^p} \leq C_{p,s} \|f\|_{L^p}.$$

Note that, by standardizing limiting arguments, taking the supremum in (2) to be over all compactly supported m such that $\|m\|_{V^r} \leq 1$ does not change the definition of \mathcal{M}_r . Applying the decomposition (6), we see that for any compactly supported m with $\|m\|_{V^r} \leq 1$ we have

$$\begin{aligned} |(m\hat{f})^\sim(x)| &\leq \sum_{j \geq 0} \sum_{v \in \Upsilon_j} |b_v(1_v \hat{f})^\sim(x)| \\ &\leq \sum_{j \geq 0} \sup_{v \in \Upsilon_j} |b_v| |\Upsilon_j|^{\frac{1}{s'}} \left(\sum_{v \in \Upsilon_j} |(1_v \hat{f})^\sim(x)|^s \right)^{1/s} \\ &\leq C_{r,s} \sup_{\mathcal{I}} \left(\sum_{I \in \mathcal{I}} |(1_I \hat{f})^\sim(x)|^s \right)^{1/s} \end{aligned}$$

where, for the last inequality, we require $s < r'$.

Provided that $r < 2$ and $p > r$ we can choose an $s < r'$ with $s > 2$ and $p > s'$, giving (7) and hence (3).

The argument of Lacey [4] for $r = 1$ follows a similar pattern, except with Marcinkiewicz's method in place of [2] and the standard Carleson-Hunt theorem in place of [6].

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